

Injectivity of the Cauchy-stress tensor along rank-one connected lines under strict rank-one convexity condition

Patrizio Neff¹ and L. Angela Mihai²

October 31, 2016

Abstract

In this note, we show that the Cauchy stress tensor σ in nonlinear elasticity is injective along rank-one connected lines provided that the constitutive law is strictly rank-one convex. This means that $\sigma(F + \xi \otimes \eta) = \sigma(F)$ implies $\xi \otimes \eta = 0$ under strict rank-one convexity. As a consequence of this seemingly unnoticed observation, it follows that rank-one convexity and a homogeneous Cauchy stress imply that the left Cauchy-Green strain is homogeneous, as is shown in [12].

Mathematics Subject Classification: 74B20, 74G65, 26B25

Key words: rank-one convexity, nonlinear elasticity, Cauchy stress tensor, invertible stress-strain law

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¹Corresponding author: Patrizio Neff, Head of Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany, email: patrizio.neff@uni-due.de

²L. Angela Mihai, Senior Lecturer in Applied Mathematics, School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, CF24 4AG, UK, email: MihaiLA@cardiff.ac.uk

1 Introduction

The search for a priori constitutive inequalities has been termed by Truesdell [20, 21] the ‘‘Hauptproblem’’ of nonlinear elasticity. These constitutive inequalities should guarantee reasonable physical response under all possible circumstances [17, 18.6.3]. We focus here on one of these requirements, namely rank-one convexity, and exhibit a hitherto unnoticed consequence of strict rank-one convexity in connection with the Cauchy stress tensor.

Following a definition by Ball [2, Definition 3.2], we say that W is *strictly rank-one convex* on $\text{GL}^+(3) = \{X \in \mathbb{R}^{3 \times 3} \mid \det X > 0\}$ if it is strictly convex on all closed line segments in $\text{GL}^+(3)$ with end points differing by a matrix of rank one, i.e.,

$$W(F + (1 - \theta)\xi \otimes \eta) < \theta W(F) + (1 - \theta)W(F + \xi \otimes \eta) \quad (1.1)$$

for all $F \in \text{GL}^+(3)$, $\theta \in [0, 1]$ and all $\xi, \eta \in \mathbb{R}^3$ with $F + t\xi \otimes \eta \in \text{GL}^+(3)$ for all $t \in [0, 1]$, where $\xi \otimes \eta$ denotes the dyadic product. Rank-one convexity is connected to the study of wave propagation [15] or hyperbolicity of the dynamical equations of elasticity, and plays an important role in the existence and uniqueness theory for linear elastostatics and elastodynamics [14, 6, 4, 16], cf. [10]. Important criteria for the rank-one convexity of stored energy density functions were first established by Knowles and Sternberg [9], see also [11, 13, 7].

In this paper we use the Frobenius tensor norm $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}}$, where $\langle X, Y \rangle_{\mathbb{R}^{n \times n}}$ is the standard Euclidean scalar product on $\mathbb{R}^{n \times n}$. If no confusion can arise, we will suppress the subscripts $\mathbb{R}^{n \times n}$. The identity tensor on $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(X) = \langle X, \mathbb{1} \rangle$.

Rank-one convexity is preferably expressed in terms of the stored energy density $W(F)$ or as a monotonicity requirement for the first Piola-Kirchhoff stress tensor $S_1 = D W(F)$ along rank-one lines, i.e.,

$$\langle S_1(F + \xi \otimes \eta) - S_1(F), \xi \otimes \eta \rangle_{\mathbb{R}^{3 \times 3}} > 0 \quad \forall \xi \otimes \eta \neq 0, \quad \forall F \in \text{GL}^+(3), \quad (1.2)$$

which, if W is twice-differentiable, turns into the well-known strong-ellipticity condition

$$D^2 W(F)(\xi \otimes \eta, \xi \otimes \eta) > 0 \quad \forall \xi \otimes \eta \neq 0, \quad \forall F \in \text{GL}^+(3). \quad (1.3)$$

Since objective stored energy density functions cannot be convex in F [18], the first Piola-Kirchhoff stress $S_1(F)$ will, in general, not be injective ([14, 6.2.38], [17, 18.4.5]). However, the strict monotonicity condition (1.2) means that $S_1(F + \xi \otimes \eta) = S_1(F)$ implies $\xi \otimes \eta = 0$. This motivates the following

Definition 1.1. *The stress tensor S is rank-one injective at F if*

$$S(F + \xi \otimes \eta) = S(F) \quad \Longleftrightarrow \quad \xi \otimes \eta = 0. \quad (1.4)$$

In this sense, if the stored energy density is strictly rank-one convex, then the first Piola-Kirchhoff stress tensor $S_1(F)$ is everywhere rank-one injective.

The only well-known consequence of rank-one convexity in connection to the Cauchy stress tensor are the Baker-Ericksen inequalities [1] for the principal values of the Cauchy stress. These, however, are meaningful only for isotropy [5].

Here, we show by a short and elementary calculation that strict monotonicity of the first Piola-Kirchhoff stress tensor S_1 along rank-one lines implies injectivity of the Cauchy stress tensor along rank-one lines.

This elementary observation answers a question raised in a recent contribution [12]: Is it impossible for a strictly rank-one convex stored energy to admit a continuous deformation that corresponds

to a homogeneous Cauchy stress field but has jumps in its deformation gradient field across planar interfaces? Indeed, in [12] we show that a non rank-one convex formulation may allow for a deformation with a homogeneous Cauchy stress field but an inhomogeneous left Cauchy-Green strain field.

We consider the following general situation: Let

$$\sigma: \text{GL}^+(3) \rightarrow \text{Sym}(3), \quad F \mapsto \sigma(F)$$

denote the Cauchy stress response function induced by the stored energy density W , and let $F \in \text{GL}^+(3)$ be such that

$$\sigma(F + \xi \otimes \eta) = \sigma(F) \quad (1.5)$$

for some $\xi \otimes \eta \neq 0$. We recall the basic relation [3]

$$\sigma(F) = S_1(F) (\text{Cof}(F))^{-1} \quad (1.6)$$

and note that in case of isotropy we may write

$$\sigma(F) = \tilde{\sigma}(F F^T) = \tilde{\sigma}(B), \quad \tilde{\sigma}: \text{Sym}^+(3) \rightarrow \text{Sym}(3), \quad B \mapsto \tilde{\sigma}(B). \quad (1.7)$$

In isotropic nonlinear elasticity, a number of energies (suitable Neo-Hooke, Mooney-Rivlin [3, 14], the exponentiated Hencky energy [13]) define an invertible Cauchy stress-strain relation, in the sense that the mapping $B \mapsto \tilde{\sigma}(B)$ is invertible. In this case $\sigma(F + \xi \otimes \eta) = \tilde{\sigma}(\widehat{B}) = \tilde{\sigma}(B) = \sigma(F)$ leads to $B = \widehat{B}$. This, together with $\det \widehat{F} = \det F > 0$ implies $\xi \otimes \eta = 0$ in (1.5). A self-contained elementary proof of this fact is given in the appendix.

Our subsequent development will be independent of any invertibility assumption for the Cauchy stress σ in the isotropic representation with $\tilde{\sigma}$.

2 Injectivity of the Cauchy-stress tensor along rank-one lines for strictly rank-one convex energies

We will show that equality (1.5) combined with strict rank-one convexity in the format of (1.2) leads to a contradiction.¹

Proof. To this aim, using (1.6) we compute

$$\sigma(F + \xi \otimes \eta) = S_1(F + \xi \otimes \eta) (\text{Cof}(F + \xi \otimes \eta))^{-1}, \quad \sigma(F) = S_1(F) (\text{Cof}(F))^{-1}. \quad (2.1)$$

Hence, from (1.5) it follows that

$$\begin{aligned} S_1(F + \xi \otimes \eta) (\text{Cof}(F + \xi \otimes \eta))^{-1} &= S_1(F) (\text{Cof}(F))^{-1} \\ \iff S_1(F + \xi \otimes \eta) &= S_1(F) (\text{Cof}(F))^{-1} (\text{Cof}(F + \xi \otimes \eta)). \end{aligned} \quad (2.2)$$

¹The following alternative proof, which uses the identity $\text{Cof}(F + \xi \otimes \eta) \cdot \eta = \text{Cof} F \cdot \eta$, see [17, eq. 1.1.18], was kindly suggested by the reviewer:

$$\begin{aligned} \langle S_1(F + \xi \otimes \eta) - S_1(F), \xi \otimes \eta \rangle &= \langle \sigma(F + \xi \otimes \eta) \cdot \text{Cof}(F + \xi \otimes \eta) - \sigma(F) \cdot \text{Cof} F, \xi \otimes \eta \rangle \\ &= \langle \sigma(F + \xi \otimes \eta), (\xi \otimes \eta) (\text{Cof}(F + \xi \otimes \eta))^T \rangle - \langle \sigma(F), (\xi \otimes \eta) (\text{Cof} F)^T \rangle \\ &= \langle \sigma(F + \xi \otimes \eta), \xi \otimes (\text{Cof}(F + \xi \otimes \eta) \cdot \eta) \rangle - \langle \sigma(F), \xi \otimes (\text{Cof} F \cdot \eta) \rangle \\ &= \langle \sigma(F + \xi \otimes \eta), \xi \otimes (\text{Cof}(F) \cdot \eta) \rangle - \langle \sigma(F), \xi \otimes (\text{Cof} F \cdot \eta) \rangle \\ &= \langle \sigma(F + \xi \otimes \eta) - \sigma(F), \xi \otimes ((\text{Cof} F) \cdot \eta) \rangle. \end{aligned}$$

If the stored energy density function is strictly rank-one convex, the latter identity implies that if $\sigma(F + \xi \otimes \eta) = \sigma(F)$, then $\xi \otimes \eta = 0$.

Since $\text{Cof}(A)\text{Cof}(B) = \text{Cof}(AB)$ and $(\text{Cof}A)^{-1} = \text{Cof}(A^{-1})$ for all $A, B \in \text{GL}^+(3)$, we obtain

$$S_1(F + \xi \otimes \eta) = S_1(F) \text{Cof}(F^{-1}F + F^{-1}\xi \otimes \eta) = S_1(F) \text{Cof}(\mathbb{1} + F^{-1}\xi \otimes \eta). \quad (2.3)$$

Using now the expansion $\text{Cof}(\mathbb{1} + H) = \text{Cof}(\mathbb{1}) + D\text{Cof}(F)\Big|_{\mathbb{1}} \cdot H + \text{Cof}(H)$, see [19], we find

$$\text{Cof}(\mathbb{1} + F^{-1}\xi \otimes \eta) = \text{Cof}(\mathbb{1}) + D\text{Cof}(F)\Big|_{\mathbb{1}} \cdot (F^{-1}\xi \otimes \eta) + \underbrace{\text{Cof}(F^{-1}\xi \otimes \eta)}_{=0}, \quad (2.4)$$

and since

$$D\text{Cof}(F) \cdot H = (\langle F^{-T}, H \rangle \mathbb{1} - F^{-T}H^T) \text{Cof}F \Rightarrow D\text{Cof}(F)\Big|_{\mathbb{1}} \cdot H = \langle \mathbb{1}, H \rangle \mathbb{1} - H^T, \quad (2.5)$$

we can rewrite equality (2.2) as

$$\begin{aligned} S_1(F + \xi \otimes \eta) &= S_1(F) \left[\mathbb{1} + D\text{Cof}(F)\Big|_{\mathbb{1}} \cdot (F^{-1}\xi \otimes \eta) \right] \\ &= S_1(F) \left[\mathbb{1} + \langle \mathbb{1}, (F^{-1}\xi \otimes \eta) \rangle \mathbb{1} - (F^{-1}\xi \otimes \eta)^T \right]. \end{aligned} \quad (2.6)$$

Going back to the strict rank-one convexity condition (1.2), we compute now

$$\begin{aligned} \langle S_1(F + \xi \otimes \eta) - S_1(F), \xi \otimes \eta \rangle &= \langle S_1(F) [\mathbb{1} + \langle \mathbb{1}, (F^{-1}\xi \otimes \eta) \rangle \mathbb{1} - (F^{-1}\xi \otimes \eta)^T] - S_1(F), \xi \otimes \eta \rangle \\ &= \langle \langle \mathbb{1}, (F^{-1}\xi \otimes \eta) \rangle S_1(F) - S_1(F) (F^{-1}\xi \otimes \eta)^T, \xi \otimes \eta \rangle \\ &= \langle \mathbb{1}, (F^{-1}\xi \otimes \eta) \rangle \langle S_1(F), \xi \otimes \eta \rangle - \langle S_1(F), (\xi \otimes \eta) (F^{-1}\xi \otimes \eta)^T \rangle \\ &= \langle \mathbb{1}, (F^{-1}\xi \otimes \eta) \rangle \langle S_1(F), \xi \otimes \eta \rangle - \langle S_1(F), \langle \eta, F^{-1}\xi \rangle (\xi \otimes \eta) \rangle \\ &= \langle \eta, F^{-1}\xi \rangle \langle S_1(F), \xi \otimes \eta \rangle - \langle \eta, F^{-1}\xi \rangle \langle S_1(F), \xi \otimes \eta \rangle = 0. \end{aligned} \quad (2.7)$$

Here, we have used that $\langle \mathbb{1}, a \otimes b \rangle_{\mathbb{R}^{3 \times 3}} = \langle b, a \rangle_{\mathbb{R}^3}$ as well as $(a \otimes b)(c \otimes d) = \langle b, c \rangle (a \otimes d)$, for all $a, b, c, d \in \mathbb{R}^3$.

Therefore, the assumption of the non-injectivity along rank-one lines (1.5) is in contradiction to the strict rank-one convexity (1.2). \square

In summary, we have shown that strict rank-one convexity implies that

$$\sigma(F + \xi \otimes \eta) = \sigma(F) \quad \text{is impossible for a nontrivial} \quad \xi \otimes \eta \neq 0, \quad \xi, \eta \in \mathbb{R}^3. \quad (2.8)$$

In these terms, we have thus proved that

$$\begin{aligned} \text{strict rank-one convexity} &\implies \text{the Cauchy stress } \sigma \\ &\text{is rank-one injective for all } F \in \text{GL}^+(3). \end{aligned} \quad (2.9)$$

3 Conclusion

Our simple calculation shows that for strictly rank-one convex stored energy density functions it is impossible to have a constant Cauchy-stress field in response to a rank-one connected laminate microstructure. Our result suggests also that some form of injectivity for the Cauchy stress is natural to require in nonlinear elasticity and this injectivity should be the object of further studies.

In order to give added perspective to our result on injectivity of the Cauchy stress, let us consider the uni-constant Blatz-Ko stored energy density function

$$W(F) = \frac{\mu}{2} \left(\|F\|^2 + \frac{2}{\det F} - 5 \right).$$

This function is strictly polyconvex, hence strictly rank-one elliptic with Cauchy stress

$$\tilde{\sigma}: \text{Sym}^+(3) \rightarrow \text{Sym}(3), \quad \tilde{\sigma}(B) = \frac{\mu}{\det B} \left(\sqrt{\det B} \cdot B - \mathbf{1} \right). \quad (3.10)$$

The Cauchy stress in (3.10) is not bijective, which can be seen along the family $B = \alpha \cdot \mathbf{1}$, $\alpha > 0$. The spherical part $\frac{1}{3} \text{tr}(\sigma)$ of the Cauchy stress first increases for increasing α and then decreases. Thus strict polyconvexity alone is not enough to prevent this unphysical response [8]. We need to require a condition beyond polyconvexity. Injectivity of the Cauchy stress is a candidate implying the classical pressure-compression inequality [13]

$$\frac{1}{3} \text{tr}(\sigma(\lambda \mathbf{1})) \cdot [\lambda - 1] > 0, \quad (3.11)$$

which would already exclude the deficiency of the Blatz-Ko strain energy.

4 Acknowledgements

The support for L. Angela Mihai by the Engineering and Physical Sciences Research Council of Great Britain under research grant EP/M011992/1 is gratefully acknowledged. We thank the reviewer for pointing out the shorter proofs noted in Section 2 and in the Appendix.

References

- [1] M. Baker and J.E. Ericksen. Inequalities restricting the form of the stress-deformation relations for isotropic elastic solids and Reiner-Rivlin fluids. *J. Wash. Acad. Sci.*, 44:33–35, 1954.
- [2] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, 63:337–403, 1977.
- [3] P.G. Ciarlet. *Three-Dimensional Elasticity*, Elsevier, Studies in Mathematics and its Applications, Amsterdam, 1988.
- [4] W. Edelman and R. Fosdick. A note on non-uniqueness in linear elasticity theory. *Z. Angew. Math. Phys.*, 19(6):906–912, 1968.
- [5] R. Fosdick and M. Silhavy. Generalized Baker-Ericksen inequalities. *J. Elasticity*, 85:39–44, 2006.
- [6] R. Fosdick, M.D. Piccioni, and G. Puglisi. A note on uniqueness in linear elastostatics. *J. Elasticity*, 88(1):79–86, 2007.
- [7] I.D. Ghiba, R.J. Martin, and P. Neff. Rank-one convexity implies polyconvexity in planar objective, isotropic and incompressible nonlinear elasticity. *submitted*, 2016.
- [8] C.S. Jog and K.D. Patil. Conditions for the onset of elastic and material instabilities in hyperelastic materials. *Archive of Applied Mechanics*, 83.5:661–684, 2013.
- [9] J.K. Knowles and E. Sternberg. On the ellipticity of the equations of nonlinear elastostatics for a special material. *J. Elasticity*, 5(3-4):341–361, 1975.
- [10] J.K. Knowles and E. Sternberg. On the failure of ellipticity of the equations for finite elastostatic plane strain. *Arch. Rat. Mech. Anal.*, 63(4):321–336, 1976.
- [11] R.J. Martin, I.D. Ghiba, and P. Neff. Rank-one convexity implies polyconvexity for isotropic, objective and isochoric elastic energies in the two-dimensional case. *to appear in Proc. Roy. Soc. Edinburgh Sect. A*, 2016.
- [12] L. A. Mihai and P. Neff. Hyperelastic bodies under homogeneous Cauchy stress induced by non-homogeneous finite deformations. *to appear in Int. J. Nonl. Mechanics*, 2016.
- [13] P. Neff, I. D. Ghiba, and J. Lankeit. The exponentiated Hencky-logarithmic strain energy. Part I: Constitutive issues and rank-one convexity. *J. Elasticity*, 121:143–234, 2015.
- [14] R.W. Ogden. *Non-Linear Elastic Deformations*. Mathematics and its Applications. Ellis Horwood, Chichester, 1. edition, 1983.
- [15] K.N. Sawyers and R. Rivlin. On the speed of propagation of waves in a deformed compressible. elastic material. *Z. Angew. Math. Phys.*, 29:245–251, 1978.

- [16] H. Simpson and S. Spector. On bifurcation in finite elasticity: buckling of a rectangular rod. *J. Elasticity*, 92(3):277–326, 2008.
- [17] M. Silhavy. The Mechanics and Thermodynamics of Continuous Media. *Springer*, 1997.
- [18] J. Schröder and P. Neff. Poly-, Quasi- and Rank-One Convexity in Applied Mechanics. *Springer*, 2010.
- [19] J. Schröder and P. Neff. Invariant formulation of hyperelastic transverse isotropy based on polyconvex free energy functions. *Int. J. Solids Struct.*, 40(2):401–445, 2003.
- [20] C. Truesdell. Das ungelöste Hauptproblem der endlichen Elastizitätstheorie. *Z. Angew. Math. Mech.*, 36: 97–103, 1956.
- [21] C. Truesdell and R. Toupin. The Classical Field Theories. In S. Flügge, editor, *Handbuch der Physik*, volume III/1. Springer, Heidelberg, 1960.

5 Appendix

In this appendix we show² that

$$\widehat{F} \widehat{F}^T = F F^T, \quad \widehat{F} = F + \xi \otimes \eta, \quad \det \widehat{F}, \det F > 0 \quad \implies \quad \xi \otimes \eta = 0. \quad (5.12)$$

We note that \widehat{F} and F are twins [17, Sect. 2.5] since they are rank-one connected and their principal stretches coincide. Here, not only their principal stretches coincide, but the left-stretch tensor is the same as well.

Proof. Since $\widehat{F} \widehat{F}^T = F F^T$, we see that $(\det \widehat{F})^2 = (\det F)^2$, and by assumption (5.12)₃ we can conclude that $\det \widehat{F} = \det F$. Since

$$\det(F + \xi \otimes \eta) = \det(F(\mathbb{1} + F^{-1} \xi \otimes \eta)) = \det F \cdot \det(\mathbb{1} + F^{-1} \xi \otimes \eta) = \det F \cdot (1 + \text{tr}(F^{-1} \xi \otimes \eta)) \quad (5.13)$$

and $\det(F + \xi \otimes \eta) = \det \widehat{F} = \det F$, by (5.12)₂ we conclude from (5.13)

$$\text{tr}(F^{-1} \xi \otimes \eta) = \langle F^{-1} \xi, \eta \rangle = 0. \quad (5.14)$$

Assumption (5.12)₁ and (5.12)₂ together imply

$$\widehat{F} \widehat{F}^T = F F^T + F \eta \otimes \xi + \xi \otimes F \eta + \|\eta\|^2 (\xi \otimes \xi) = F F^T, \quad (5.15)$$

thus we must have

$$F \eta \otimes \xi + \xi \otimes F \eta + \|\eta\|^2 (\xi \otimes \xi) = 0. \quad (5.16)$$

We introduce $\widehat{\xi} = F^{-1} \xi$, $\xi = F \widehat{\xi}$ and insert into (5.14) and (5.16) to yield

$$F \eta \otimes F \widehat{\xi} + F \widehat{\xi} \otimes F \eta + \|\eta\|^2 (F \widehat{\xi} \otimes F \widehat{\xi}) = 0, \quad \langle \widehat{\xi}, \eta \rangle = 0. \quad (5.17)$$

This is equivalent to

$$F \{ \eta \otimes \widehat{\xi} + \widehat{\xi} \otimes \eta + \|\eta\|^2 (\widehat{\xi} \otimes \widehat{\xi}) \} F^T = 0, \quad \langle \widehat{\xi}, \eta \rangle = 0. \quad (5.18)$$

Since $\det F > 0$ we have as well

$$\eta \otimes \widehat{\xi} + \widehat{\xi} \otimes \eta + \|\eta\|^2 (\widehat{\xi} \otimes \widehat{\xi}) = 0, \quad \langle \widehat{\xi}, \eta \rangle = 0. \quad (5.19)$$

Multiplying (5.19) with $\eta \neq 0$ we obtain $\underbrace{\eta \langle \widehat{\xi}, \eta \rangle}_{=0} + \widehat{\xi} \|\eta\|^2 + \|\eta\|^2 \widehat{\xi} \underbrace{\langle \widehat{\xi}, \eta \rangle}_{=0} = 0$. Hence, $\widehat{\xi} \|\eta\|^2 = 0$ implies $\widehat{\xi} = 0$. \square

²The following alternative proof was kindly suggested by the reviewer: Rewriting (5.16) as

$$(F \eta + \|\eta\|^2 \xi) \otimes \xi = -\xi \otimes F \eta$$

and recalling that $a \otimes b = c \otimes d \neq 0$ if and only if there is $\lambda \in \mathbb{R} \setminus \{0\}$ such that $a = \lambda c$ and $b = \frac{d}{\lambda}$, then the assumption $\xi \otimes \eta \neq 0$ implies

$$F \eta + \|\eta\|^2 \xi = \lambda \xi, \quad F \eta = -\lambda \xi,$$

and thus $\lambda = \frac{1}{2} \|\eta\|^2$. But then

$$\det(F + \xi \otimes \eta) = \det F + \langle \text{Cof}(F) \eta, \xi \rangle = (1 + \langle \eta, F^{-1} \xi \rangle) \det F = \left(1 - \langle \eta, \frac{2}{\|\eta\|^2} \eta \rangle\right) \det F = -\det F,$$

which contradicts the assumption $F, F + \xi \otimes \eta \in \text{GL}^+(3)$.